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Differential Calculi on the Quantum Superplane

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Using some natural conditions less restrictive than the $GL_{q_{ij},s}(m/n)$ invariance, we present two possible multiparametric differential calculi on the quantum superplane. We show that there exists a new differential calculus which is different from the known one, generalizing the Wess–Zumino formalism to the superspace case. We discuss some *-algebra structures leaving invariant this differential calculus. The (1 + 1)-dimensional case is analyzed and a realization of the super-Virasoro algebra on this particular quantum superspace is given.

1. INTRODUCTION

The theory of quantum groups (Drinfel'd, 1985, 1986; Kulish and Reshetikhin, 1983; Jimbo, 1985, 1986; Corrigan *et al.*, 1990; Sudberg, 1990) has emerged in the last few years as a nontrivial generalization of the notion of Lie groups. The latter are recovered in the limit $q \to 1$, where q is a continuous deformation parameter (or a set of parameters). The quantum groups were realized on the quantum hyperplane, in which coordinates are noncommuting (Manin, 1989; Takhtajan, 1989; Doebner and Henning, 1989). Woronowicz (1989) showed that these structures provide a concrete example of noncommutative differential geometry. Wess and Zumino (1990) developed a simpler example of noncommutative geometry. They proved that one can introduce a consistent, GL_q-covariant differential calculus on the noncommutative space of the quantum hyperplane. This scheme was generalized to the multiparametric case (Schirrmacher, 1991; Soni, 1991a) and also extended to the quantum superplane (Soni, 1991b,c; Kobayashi and Uematsu, 1992).

Brzezinski *et al.* (1992) gave another generalization of the work of Wess and Zumino, based on the covariance constraint; they assumed that the differential calculus is invariant only under the scale transformations instead

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of with respect to the entire GL_q quantum group. Under a such condition they showed that there exist two families of possible and nonequivalent multiparametric differential calculi. One of them coincides with the differential calculus in Schirrmacher (1991) and Soni (1991a).

In the present work we generalize the construction given in Brzezinski *et al.* (1992) to the superplane case. Based on similar assumptions applied to the superplane, we show that there exist two families of possible and nonequivalent differential schemes. The first one (I) has more deformation parameters (q_{IJ}, p_i, s), while the second one (II) has only (q_{IJ}, s) as deformation parameters. These two families intersect for $s = p_i = 1$ and also when we eliminate the bosonic part in both cases. Noting that case II coincides with that given in Soni (1991c), we are interested in this paper in examining explicitly the first one. By considering some conditions on the deformation parameters we will discuss the existence of some possible involutions leaving this new differential structure invariant on the quantum superplane. Then we study the $(1 + 1)$-dimensional case, which leads to a realization of the deformed super-Virasoro algebra.

This paper is organized as follows: In the Section 2 we give the possible multiparametric differential calculi on the quantum superplane. Section 3 is devoted to the examination of the different antilinear antiinvolutions. The last section treats the realization of the deformed super-Virasoro algebra.

2. POSSIBLE MULTIPARAMETRIC DIFFERENTIAL CALCULI ON THE QUANTUM SUPERPLANE

In this section we introduce the differential calculus on the quantum superplane which obeys the following conditions:

(i) The linear differential operator d is nilpotent.

(ii) d satisfies the graded Leibniz rule.

(iii) The differential calculus is invariant under the transformations

$$x^i \to \alpha^i x^i, \qquad i = 1, \ldots, m$$

$$\theta^\alpha \to T^\alpha_\beta \theta^\beta, \qquad \alpha, \beta = m + 1, \ldots, m + n$$

where the elements x^i and θ^α denote the bosonic and fermionic, respectively (Grassmannian) generators of the $(m + n)$-dimensional quantum superplane.

The last condition is less restrictive than the invariance under $GL_{q_{IJ},s}(m/n)$ required in Soni (1991b,c) and Kobayashi and Uematsu (1992). Its interpretation in the language of partial derivatives, which was given in Brzezinski *et al.* (1992) for the quantum plane, remains valid for the present case of the superplane. Indeed, let us call the i bosonic partial derivatives and the α fermionic partial derivative of the function $f(\ldots) \equiv f(x^1, \ldots, x^m,$

$\theta^{m+1}, \ldots, \theta^{m+n}$), respectively, the functions $\partial_i f(x^1, \ldots, x^m, \theta^{m+1}, \ldots, \theta^{m+n})$ and $\partial_\alpha f(x^1, \ldots, x^m, \theta^{m+1}, \ldots, \theta^{m+n})$ such that

$$df(\ldots) = \sum_{i=1}^{m} dx^i\, \partial_i f(\ldots) + \sum_{\alpha=m+1}^{m+n} d\theta^\alpha\, \partial_\alpha f(\ldots)$$

The condition (iii) means that if the function f has the form

$$f(x^1, \ldots, x^m, \theta^{m+1}, \ldots, \theta^{m+n}) = (x^i)^m g(x^1, \ldots, \hat{x}^i, \ldots, x^m, \theta^{m+1}, \ldots, \theta^{m+n})$$

then

$$(\partial_i)^{m+1} f = 0$$

The differential calculus (Wess and Zumino, 1990) is defined in terms of matrices B and C [$B, C \in \mathrm{End}(\mathbf{C}^{m+n} \otimes \mathbf{C}^{m+n})$] satisfying

$$x^I x^J - (-1)^{\hat{I}\hat{J}} B^{IJ}_{KL} x^K x^L = 0 \tag{1a}$$

$$x^I dx^J - (-1)^{\hat{I}(\hat{J}+1)} C^{IJ}_{KL} dx^K x^L = 0 \tag{1b}$$

where $I, J = 1, \ldots, m + n$, and the supernumerary generators x^I denote the compact form of the pair (x^i, θ^α) of bosonic and fermionic coordinates such that $\hat{I} = 0$ for $I = 1, \ldots, m$ and $\hat{I} = 1$ for $I = m + 1, \ldots, m + n$. Note that any power in term of $\hat{I}$ is mod 2.

In the language of matrices B and C, associativity and consistency with conditions (i)–(iii) require that B and C are subject of the following constraints:

$$\hat{B}_{23}\hat{B}_{13}\hat{B}_{12} = \hat{B}_{12}\hat{B}_{13}\hat{B}_{23} \tag{2a}$$

$$\tilde{C}_{23}\tilde{C}_{13}\tilde{C}_{12} = \tilde{C}_{12}\tilde{C}_{13}\tilde{C}_{23} \tag{2b}$$

$$(\delta^I_K \delta^J_L - (-1)^{\hat{I}\hat{J}} B^{IJ}_{KL})(\delta^K_M \delta^L_N + (-1)^{\hat{K}\hat{L}} C^{KL}_{MN}) = 0 \tag{2c}$$

$$\hat{B}_{23}\hat{C}_{13}\hat{C}_{12} = \hat{C}_{12}\hat{C}_{13}\hat{B}_{23} \tag{2d}$$

where $\hat{B}^{IJ}_{KL} \equiv (-1)^{\hat{I}\hat{J}} B^{JI}_{KL}$ (analogously for C), $\tilde{C}^{IJ}_{KL} = (-1)^{\hat{J}+\hat{K}}(-1)^{\hat{I}\hat{J}} C^{JI}_{KL}$, and the subscripts 1, 2, and 3 refer to different couples of indices; thus the repeated ones mean the usual matrix multiplication. In an appropriate basis, the generalization to the case of the quantum superplane from Manin's plane leads to the fact that the matrix B should be such that

$$x^I x^J - (-1)^{\hat{I}\hat{J}} q_{IJ} x^J x^I = 0 \tag{3}$$

or again (in the above convention)

$$x^i x^j = q_{ij} x^j x^i, \qquad x^i \theta^\alpha = q_{i\alpha} \theta^\alpha x^i, \qquad \theta^\alpha \theta^\beta = -q_{\alpha\beta} \theta^\beta \theta^\alpha, \qquad (\theta^\alpha)^2 = 0$$

where the q's are arbitrary complex parameters.

The most general matrix B satisfying (2a) and (3) has the form

$$B = \sum_{I} \left(\frac{1}{s}\right)^{\hat{I}} e_I^I \otimes e_I^I + \sum_{I<J} \frac{q_{IJ}}{s} e_J^I \otimes e_I^J + \sum_{I>J} q_{IJ} e_J^I \otimes e_I^J + \left(1 - \frac{1}{s}\right) \sum_{I<J} (-1)^{\hat{I}\hat{J}} e_I^I \otimes e_J^J \tag{4}$$

where $e_J^I \in \mathrm{Mat}(C^{m+n})$ are matrices with single nonzero element (equal to 1) at (I, J) and $s \in C$.

The question of finding possible differential calculi on the quantum superplane is equivalent to searching for a matrix C satisfying equations (2b)–(2d) consistent with conditions (i)–(iii). To this end, let us recall that in the case of the quantum plane, this search led to two possible distinct forms of C (Brzezinski *et al.*, 1992). To summarize, the matrices B and all possible C's were given explicitly as follows:

$$B = \sum_{i} e_i^i \otimes e_i^i + \sum_{i<j} \frac{q_{ij}}{s} e_j^i \otimes e_i^j + \sum_{i>j} q_{ij} e_j^i \otimes e_i^j + \left(1 - \frac{1}{s}\right) \sum_{i<j} e_i^i \otimes e_j^j \tag{5}$$

$$C_{\mathrm{I}} = \sum_{i} p_i e_i^i \otimes e_i^i + \sum_{i\neq j} q_{ij} e_j^i \otimes e_i^j \tag{6a}$$

$$C_{\mathrm{II}} = sB \tag{6b}$$

The generalization of this scheme to the quantum superplane gives, in connection with the matrix B of equation (4), two different and distinct forms of the matrix C:

$$C_{\mathrm{I}} = \sum_{I} (p_{\mathrm{I}})^{\hat{I}+1} e_I^I \otimes e_I^I + \sum_{I<J} q_{IJ} e_J^I \otimes e_I^J + \sum_{I>J} (s)^{\hat{I}\hat{J}} q_{IJ} e_J^I \otimes e_I^J + \sum_{I<J} (-1)^{\hat{I}\hat{J}} [(s)^{\hat{I}\hat{J}} - 1] e_I^I \otimes e_J^J, \qquad p_I \in C \tag{7a}$$

$$C_{\mathrm{II}} = sB \tag{7b}$$

These forms of C, that is, C_{I} and C_{II} [equations (7a) and (7b)], are the supersymmetric versions of equations (6a) and (6b), respectively. Moreover, notice that if we eliminate the bosonic part from the quantum superplane (i.e., the quantum superplane contains only the fermionic coordinates), we obtain

$$C_{\mathrm{I}} = C_{\mathrm{II}} = \sum_{\alpha} e_\alpha^\alpha \otimes e_\alpha^\alpha + \sum_{\alpha>\beta} s q_{\alpha\beta} e_\beta^\alpha \otimes e_\alpha^\beta + \sum_{\alpha<\beta} q_{\alpha\beta} e_\beta^\alpha \otimes e_\alpha^\beta - (s - 1) \sum_{\alpha<\beta} e_\alpha^\alpha \otimes e_\beta^\beta \tag{8}$$

We point out that the second form of C, equation (7b), coincides with the one given in Soni (1991c) and Kobayashi and Uematsu (1992), so we will be interested in what follows in the study of the new differential calculus corresponding to the matrix C_{I}, equation (7a). The explicit form of this differential calculus is then, for $I < J$,

$$\begin{aligned}
x\text{-}dx: \quad & x^I\, dx^J = (-1)^{\hat{I}(\hat{J}+1)} q_{IJ}\, dx^J\, x^I + (-1)^{\hat{I}}[(s)^{\hat{I}\hat{J}} - 1]\, dx^I\, x^J \\
& x^J\, dx^I = (-1)^{\hat{J}(\hat{I}+1)}(s)^{\hat{I}\hat{J}} q_{JI}\, dx^I\, x^J \\
& x^I\, dx^I = (p_I)^{\hat{I}+1}\, dx^I\, x^I
\end{aligned} \tag{9}$$

All other commutation relations which complete the scheme of this differential calculus can be deduced from the technique used by Wess and Zumino (1990). We list them as follows for $I < J$:

$$dx\text{-}dx: \quad dx^I\, dx^I = 0 \qquad \text{for} \quad \hat{I} = 0 \tag{10}$$

$$dx^I\, dx^J = (-1)^{(\hat{I}+1)(\hat{J}+1)} \frac{q_{IJ}}{(s)^{\hat{I}\hat{J}}}\, dx^J\, dx^I$$

$$\begin{aligned}
\partial\text{-}x: \quad & \partial_I x^J = (-1)^{\hat{I}\hat{J}} \frac{(s)^{\hat{I}\hat{J}}}{q_{IJ}}\, x^J\, \partial_I \\
& \partial_J x^I = (-1)^{\hat{I}\hat{J}} q_{IJ} x^I\, \partial_J \\
& \partial_I x^I = 1 + (-1)^{\hat{I}}(p_I)^{\hat{I}+1} x^I\, \partial_I + (-1)^{\hat{I}} \sum_{J>I} (-1)^{\hat{J}}[(s)^{\hat{I}\hat{J}} - 1] x^J\, \partial_J
\end{aligned} \tag{11}$$

$$\begin{aligned}
\partial\text{-}dx: \quad & \partial_I\, dx^J = (-1)^{\hat{I}(\hat{J}+1)} \frac{1}{q_{IJ}}\, dx^J\, \partial_I \\
& \partial_J\, dx^I = (-1)^{\hat{J}(\hat{I}+1)} \frac{q_{IJ}}{(s)^{\hat{I}\hat{J}}}\, dx^I\, \partial_J \\
& \partial_I\, dx^I = \left(\frac{1}{p_I}\right)^{\hat{I}+1} dx^I\, \partial_I + (-1)^{\hat{I}} \sum_{J>I} \left(\frac{1}{(s)^{\hat{I}\hat{J}}} - 1\right) dx^J\, \partial_J
\end{aligned} \tag{12}$$

$$\partial\text{-}\partial: \quad \partial_I \partial_I = 0 \qquad \text{for} \quad \hat{I} = 1 \tag{13}$$

$$\partial_I \partial_J = (-1)^{\hat{I}\hat{J}} \frac{q_{IJ}}{(s)^{\hat{I}\hat{J}}}\, \partial_J \partial_I$$

$$\partial\text{-}d: \quad \partial_I d = (-1)^{\hat{I}}\, d\partial_I + (-1)^{\hat{I}} \sum_J \left(\frac{1}{(s)^{\hat{I}\hat{J}}} - 1\right) dx^J\, \partial_J \partial_I \tag{14}$$

Using all the above relations, it is easy to check that $d^2 = 0$, which is consistent with our assumption.

If we consider the quantum linear transformations $(T^I_J)_{(I,J=1,\ldots,m+n)}$ on the variables and differentials, i.e., $x^I \to T^I_J x^J$ and $dx^I \to T^I_J\, dx^J$, the discussion of the invariance of this differential structure under these transformations leads to consider two distinct possibilities. If $s = p_i = 1$, $\forall i = 1, \ldots, m$, this structure is equal to that given in Soni (1991c) with $s = 1$ and the set of these transformations is just the quantum supergroup $GL_{q_{IJ},1}(m/n)$. Otherwise this structure is invariant, as we noticed above, under the action of the quantum sub-supergroup of $GL_{q_{IJ},s}(m/n)$ such that

$$T^\alpha_i = T^i_\alpha = 0 \qquad i = 1, \ldots, m; \quad \alpha = m + 1, \ldots, m + n \tag{15}$$

$$T^i_j = 0 \qquad \text{for} \quad i \neq j$$

3. THE *-ALGEBRA STRUCTURE

Now let us discuss the construction of some possible involutions leaving invariant the above new differential calculus on the quantum superplane described by the matrices B and C_I of equations (4) and (7a), respectively. We can distinguish three different possibilities.

Case 1. The antilinear antiinvolution

$$(x^i)^* = x^{m+1-i} \equiv x^{i'}, \quad i' = 1, \ldots, m + 1 \tag{16a}$$

$$(\theta^\alpha)^* = \theta^{2m+n+1-\alpha} \equiv \theta^{\alpha'}, \qquad \alpha' = 2m + n + 1 - \alpha$$

exists iff

$$s = 1, \qquad q^*_{ij} q_{i'j'} = 1, \qquad p^*_i p_{i'} = 1, \qquad q^*_{i\alpha'} = 1, \qquad q^*_{\alpha\beta} q_{\alpha'\beta'} = 1 \tag{16b}$$

In compact form, if we take the permutation σ such that

$$\sigma: \begin{pmatrix} 1 \;\ldots\ldots\; m, m + 1 \;\ldots\ldots\; m + n \\ m \;\ldots\ldots\; 1, m + n \;\ldots\ldots\; m + 1 \end{pmatrix}$$

conditions (16b) become

$$s = 1, \qquad q^*_{IJ} q_{\sigma(I)\sigma(J)} = 1, \qquad p^*_I p_{\sigma(I)} = 1 \tag{17}$$

The conjugates of differentials and derivatives are expressed by

$$(dx^I)^* = (-1)^{\widehat{\sigma(I)}}\, dx^{\sigma(I)}, \qquad (\partial_I)^* = (-p_{\sigma(I)})^{1-\widehat{\sigma(I)}} \partial_{\sigma(I)}, \qquad \widehat{\sigma(I)} = \hat{I} \tag{18}$$

Case 2. Another antilinear antiinvolution given by

$$(x^i)^* = x^i, \qquad i = 1, \ldots, m + 1 \tag{19a}$$

$$(\theta^\alpha)^* = \theta^{2m+n+1-\alpha} \equiv \theta^{\alpha'}, \qquad \alpha' = 2m + n + 1 - \alpha$$

can be considered iff

$$s = 1, \qquad q_{ij}^* q_{ij} = 1, \qquad p_i^* p_i = 1, \qquad q_{i\alpha}^* q_{i\alpha'} = 1, \qquad q_{\alpha\beta}^* q_{\alpha'\beta'} = 1 \tag{19b}$$

In compact form, equalities (19b) become

$$s = 1, \qquad q_{IJ}^* q_{\tau(I)\tau(J)} = 1, \qquad p_I^* p_{\tau(I)} = 1 \tag{20}$$

where τ is a permutation acting only on the Grassmanian part and given by

$$\tau: \begin{pmatrix} 1 \ \ldots\ldots \ m, m+1 \ \ldots\ldots \ m+n \\ 1 \ \ldots\ldots \ m, m+n \ \ldots\ldots \ m+1 \end{pmatrix}$$

The conjugates of differentials and derivatives are

$$(dx^I)^* = (-1)^{\hat{\tau(I)}} dx^{\tau(I)}, \qquad (\partial_I)^* = (-p_{\tau(I)})^{1-\hat{\tau(I)}} \partial_{\tau(I)} \tag{21}$$

Case 3. A last possible antilinear antiinvolution is described by

$$(x^i)^* = x^i, \qquad (\theta^\alpha)^* = \theta^\alpha \tag{22}$$

It exists if the following constraints on the deformation parameters hold:

$$s^* s = 1, \qquad q_{IJ}^* q_{IJ} = 1, \qquad p_I^* p_I = 1$$

This involution acts on differentials and derivatives as

$$(dx^I)^* = (-1)^{\hat{I}} dx^I, \qquad (\partial_I)^* = (-p_I)^{1-\hat{I}} (S^{I-n-m})^{\hat{I}} \partial_I$$

We notice that, by a direct computation, one can show that the differential calculus described by equations (7a) remains invariant under these *-operations.

These involutions, especially the first case, will make it possible to introduce creation and annihilation superoperators and to construct the corresponding superalgebra. Details of this idea will be given in a further work.

4. THE (1 + 1)-DIMENSIONAL CASE AND A REALIZATION OF THE SUPER-VIRASORO ALGEBRA

This section is devoted to the study of the simpler example of the (1 + 1) quantum superplane. This particular case allows us to realize the deformed super-Virasoro algebra. Let us first set the commutation relations which interfere in the realization of this super-Virasoro algebra on the (1 + 1) quantum superplane. They are obtained immediately from the formulas of the previous section; thus the variables x and θ and their respective partial derivatives satisfy

$$
\begin{aligned}
x\theta &= q\theta x, \qquad \theta^2 = 0 \\
\partial_x \theta &= q^{-1}\theta\, \partial_x \\
\partial_x x &= 1 + px\, \partial_x \\
\partial_\theta \theta &= 1 - \theta\, \partial_\theta \\
\partial_x \partial_\theta &= q\, \partial_\theta \partial_x, \qquad \partial_\theta{}^2 = 0
\end{aligned} \tag{23}
$$

Using this differential structure, one can get a deformation of the classical super-Virasoro algebra realized on the classical (1 + 1) superplane as follows:

$$L_n = -x^{n+1}\, \partial_x, \qquad G_n = x^{n+1}\theta\, \partial_x, \qquad K_n = x^n\theta\, \partial_\theta \tag{24}$$

where L_n, G_n, and K_n ($n \in Z$) are the generators satisfying

$$
\begin{aligned}
[L_m, L_n] &= (m - n)L_{m+n}, \qquad & [K_m, G_n] &= G_{m+n} \\
[L_m, K_n] &= -nK_{m+n}, \qquad & [K_m, K_n] &= 0 \\
[L_m, G_n] &= (m - n)G_{m+n}, \qquad & \{G_m, G_n\} &= 0
\end{aligned} \tag{25}
$$

Returning to the quantum case, one can verify, owing to equations (23), the following equalities:

$$
\begin{aligned}
\partial_x x^n &= \frac{1 - p^n}{1 - p}\, x^{n-1} + p^n x^n\, \partial_x \\
\partial_\theta x^n &= q^n x^n\, \partial_\theta \\
x^n\theta &= q^n\theta\, x^n
\end{aligned} \tag{26}
$$

The deformed super-Virasoro algebra is realized by

$$L_n = -\sqrt{p}\, x^{n+1}\, \partial_x, \qquad G_n = x^{n+1}\theta\, \partial_x, \qquad K_n = x^n\theta\, \partial_\theta \qquad (n \geq -1) \tag{27}$$

A direct computation leads to the commutation relations

$$
\begin{aligned}
[L_m, L_n]_{(p^{(m-n)/2}, p^{(n-m)/2})} &= [m - n]_{p^{1/2}} L_{m+n} \\
[L_m, K_n]_{(p^{-n/2}, p^{n/2})} &= -[n]_{p^{1/2}} K_{m+n} \\
[L_m, G_n]_{(q^{-m/2}p^{(m-n)/2}, q^{m/2}p^{(n-m)/2})} &= q^{-m/2}[m - n]_{p^{1/2}} G_{m+n} \\
[K_m, K_n] &= 0, \qquad \{G_m, G_n\} = 0 \\
[K_m, G_n] &= G_{m+n}
\end{aligned} \tag{28}
$$

where

$$[x]_\alpha \equiv \frac{\alpha^x - \alpha^{-x}}{\alpha - \alpha^{-1}} \quad \text{and} \quad [A, B]_{(\alpha_1,\alpha_2)} \equiv \alpha_1 AB - \alpha_2 BA$$

5. CONCLUSION

In this paper we discussed the possible differential calculi on the quantum superplane. Using a natural assumption that is less restrictive than the known covariance under $GL_{q/j,s}(m/n)$, we showed the existence of a new differential calculus on the quantum superplane. This new scheme allows us to realize a deformed super-Virasoro algebra. The latter can be seen as a supersymmetric version of the deformed Virasoro algebra constructed (El Hassouni *et al.*, 1995) on the quantum plane. We also discussed some interesting involutions leaving this differential calculus invariant; some of them (case 2) will be interesting in the study of a deformed superspace when its bosonic coordinates are real but its fermionic ones are complex.

REFERENCES

Brzezinski, T., Dobrowski, H., and Rembielinski, J. (1992). *Journal of Mathematical Physics*, **33**, 19.

Corrigan, E., Fairlie, D. F., and Fletcher, P. (1990). *Journal of Mathematical Physics*, **31**, 776.

Doebner, H.-D., and Henning, J.-D., eds. (1989). *Quantum Groups*, Springer, Berlin.

Drinfel'd, V. S. (1985). *Soviet Mathematics-Doklady*, **32**, 254.

Drinvel'd, V. V. (1986). In *Proceedings of the International Congress of Mathematicians*, Berkeley, Vol. 1, p. 798.

El Hassouni, A., Hassouni, Y., and Tahri, E. H. (1995). LPT-ICAC preprint, Rabat.

Jimbo, M. (1985). *Letters in Mathematical Physics*, **10**, 63.

Jimbo, M. (1986). *Letters in Mathematical Physics*, **11**, 247.

Kobayashi, T., and Uematsu, T. (1992). *Zeitschrift für Physik C: Particles and Fields*, **56**, 193.

Kulish, P., and Reshetikhin, N. (1983). *Journal of Soviet Mathematics*, **23**, 2435.

Manin, Yu. I. (1989). *Communications in Mathematical Physics*, **123**, 163.

Schirrmacher, A. (1991). *Zeitschrift für Physik C: Particles and Fields*, **50**, 321.

Soni, S. K. (1991a). *Journal of Physics A: Mathematical and General*, **24**, L169.

Soni, S. K. (1991b). *Journal of Physics A: Mathematical and General*, **24**, 619.

Soni, S. K. (1991c). *Journal of Physics A: Mathematical and General*, **24**, L459.

Sudbery, A. (1990). Non-commuting coordinates and differential operators, University of York preprint.

Takhtajan, L. A. (1989). *Advanced Studies in Pure Mathematics*, **19**, 435.

Wess, J., and Zumino, B. (1990). *Nuclear Physics B (Proceedings Supplement)*, **18B**, 302.

Woronowicz, S. L. (1989). *Communications in Mathematical Physics*, **122**, 125.